Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis -I

Final Examination | Date : 29 Dec 2021
Maximum marks: 100 | Time: 3 hours
Instructor: B V Rajarama Bhat

(1) Let M the collection of all **integer** sequences $\{a_n : n \in \mathbb{N}\}$, $a_n \in \mathbb{Z}$, such that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Show that M is countable. [15]

(2) Consider the unit square $U = [0,1] \times [0,1] = \{(x,y) : 0 \leq x, y \leq 1\}$. A sequence $\{(x_n, y_n) : n \in \mathbb{N}\}$ of elements in U, is said to converge to an element (a,b) in U if

$$\lim_{n\to\infty} (\max\{|x_n - a|, |y_n - b|\}) = 0.$$

(i) Show that a sequence of elements $\{(x_n, y_n) : n \in \mathbb{N}\}$ in U converges to (a,b) if and only if $\{x_n : n \in \mathbb{N}\}$ converges to a and $\{y_n : n \in \mathbb{N}\}$ converges to b.
(ii) Show that every sequence of elements $\{(x_n, y_n) : n \in \mathbb{N}\}$ in U has a convergent subsequence. [15]

(3) Determine limsup and liminf of the following sequences of real numbers.
(i)

$$\{(-1)^n 5 + (-\frac{1}{2})^n .7 : n \in \mathbb{N}\}.$$

(ii)

$$\{\frac{n}{2^n} - \frac{n}{3^n} : n \in \mathbb{N}\}.$$

(iii)

$$\{\frac{n+6}{n^2 - 2n - 8} : n \in \mathbb{N}\}.$$

[15]

(4) Let $\{a_n : n \in \mathbb{N}\}$ be a sequence of real numbers such that $R := \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists. Fix a real number $c \in \mathbb{R}$.
(i) Show that if $R \neq 0$, the series

$$\sum_{n=1}^{\infty} a_n (x-c)^n$$

converges absolutely for $x \in (c - \frac{1}{R},\ c + \frac{1}{R})$.
(ii) State and prove a result similar to (i), when $R = 0$.
[15]

(5) Show that a continuous function $g : [0,1] \to (0,1)$ can not be surjective. Give an example of a surjective continuous function $h : (0,1) \to [0,1]$. [15]

(6) Let $k : [0,1] \to [0,1]$ be a continuous function. Show that there exists $t \in [0,1]$ such that $k(t) = 1 - t^2$. [15]

(7) Let $f : [a,b] \to \mathbb{R}$ be a function. Fix $x_0 \in [a,b]$. Assume $f^{(1)}(x_0), f^{(2)}(x_0), \ldots, f^{(n)}(x_0)$ exist. Determine the unique polynomial of degree at most n, satisfying

$$f^{(k)}(x_0) = P_n^{(k)}(x_0),\ 0 \leq k \leq n.$$

(Carefully write down your claims and proofs.) [15]